

# On structural physical approximations and entanglement breaking maps

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**Abstract.** Very recently a conjecture saying that the so-called structural physical approximations (SPA) to optimal positive maps (optimal entanglement witnesses) give entanglement breaking (EB) maps (separable states) has been posed [J. K. Korbicz *et al.*, Phys. Rev. A **78**, 062105 (2008)]. The main purpose of this contribution is to explore this subject. First, we extend the set of entanglement witnesses (EWs) supporting the conjecture. Then, we ask if SPAs constructed from other than the depolarizing channel maps also lead to EB maps and show that in general this is not the case. On the other hand, we prove an interesting fact that for any positive map  $\Lambda$  there exists an EB channel  $\Phi$  such that the SPA of  $\Lambda$  constructed with the aid of  $\Phi$  is again an EB channel. Finally, we ask similar questions in the case of continuous variable systems. We provide a simple way of construction of SPA and prove that in the case of the transposition map it gives EB channel.

## 1. Introduction

Entanglement in composite quantum systems is a crucial resource in the field of quantum information [1, 2]. It allows to realize tasks which are not achievable in classical physics with quantum teleportation [3] being the most prominent example.

It is thus natural that so much effort has been put so far in developing various methods of detection of entanglement in composite quantum systems [4]. Among others, probably the most powerful is the one based on the notion of positive maps [5, 6, 7]. It states that a given state  $\varrho$  acting on  $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$  is separable, i.e., it admits the decomposition [8]:

$$\varrho = \sum_i p_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|, \quad \sum_i p_i = 1, \quad (1)$$

if and only if the Hermitian operator  $(I \otimes \Lambda)(\varrho)$  has a nonnegative spectrum for any positive map  $\Lambda : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ . Despite its proven efficiency in entanglement detection, the above criterion is not directly applicable in experiments. This is an immediate consequence of the fact that generically positive, but not completely positive (CP) maps do not represent physical processes. It was then of a great importance to design methods which could make the experimental detection of entanglement with the aid of positive maps possible. So far, several such methods (more or less direct) have been developed [9, 10, 11, 12, 13, 14, 15].

The earliest one, formulated and investigated in a series of papers [9, 10, 11], is the so-called *structural physical approximation* (SPA). It bases on the fact that any linear map (not necessarily positive) when mixed with sufficiently large amount of the completely depolarizing channel  $D(\varrho) = \text{Tr}(\varrho) \mathbb{1}_m/m$ , which certainly is an element of the interior of the convex set of CP maps, will result in a CP map. Since the latter already represents some physical process, it can in principle be implemented experimentally. Moreover, when applied to partial maps, i.e., maps of the form  $I \otimes \Lambda$ , the method produces CP maps that hold the structure of the initial map in the sense that one can easily recover the spectrum of  $(I \otimes \Lambda)(\varrho)$  knowing the spectrum of its SPA applied to  $\varrho$ . Let us mention that a different approach to the approximation of the transposition map by some CP map was presented in Ref. [12]. Under the assumption that such approximation should work equally well for all pure states, the authors derived the best CP approximation (in terms of fidelity) of the transposition map. Interestingly, the obtained map matches SPA of the transposition map. Eventually, it should be noticed that very recently SPA of the transposition map was implemented experimentally [16].

It was observed for the first time by Fiurášek [17] that SPA of the partial transposition map  $I \otimes T$  in the two-qubit case is an entanglement breaking (EB) channel. This is a very interesting observation meaning that the implementation of the SPA of partial transposition reduces to generalized quantum measurement followed by a preparation of a suitable quantum state conditioned on the measurement outcomes [18]. Very recently, this property was extensively investigated in Ref. [19], where it was observed for many examples of optimal positive maps<sup>‡</sup> that their SPAs are entanglement breaking. Supported by the numerous examples, the authors formulated the conjecture that SPAs to optimal positive maps are EB. It should be emphasized

<sup>‡</sup> Notice that the authors investigated SPAs of positive maps rather than the "extended" maps  $I \otimes \Lambda$ , which generally do not have to be positive.

that in a series of papers [20, 21] this conjecture was confirmed for new classes of optimal entanglement witnesses (OEWS).

Due to its transparent relation to the geometry of the convex sets of quantum states with positive partial transposition and sets of entanglement witnesses [19], it is clear that despite its physical meaning, the conjecture gains also in importance from the mathematical point of view. It is then clear that its complete proof is of a great interest. The main purpose of the present paper is to address several issues related to this subject.

The paper is organized as follows. In Sec. 2 all the notions necessary for further studies are recalled. Sec. 3 contains new examples of EWs supporting the conjecture. Then, in Sec. 4 we discuss several issues related to the conjecture. In particular, we prove for any positive map  $\Lambda$  there exists an EB map  $\Phi$  such that SPA of  $\Lambda$  constructed using  $\Phi$  is EB. Also, we propose a method of construction of SPAs in continuous variable systems, and show that the SPA of the transposition map constructed with this method gives EB channel. The paper is concluded with Sec. 5.

## 2. Background and the Conjecture

First let us devote few lines to recall basic notions that will be used frequently throughout the paper. In what follows, by  $\mathcal{H}_{mn}$ ,  $\mathcal{H}_m$ , and  $M_{m,n}$  will be denoting  $\mathbb{C}^m \otimes \mathbb{C}^n$ ,  $\mathbb{C}^m \otimes \mathbb{C}^m$ ,  $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ , respectively. Then,  $\mathcal{D}_{m,n}$  denote the set of all density matrices acting on  $\mathcal{H}_{m,n}$ , i.e., positive matrices from  $M_{m,n}$  with unit trace. Finally,  $\|\cdot\|_2$  will stand for the Hilbert-Schmidt norm  $\|A\|_2 = \sqrt{\text{Tr}(A^\dagger A)}$ .

### 2.1. Positive and completely positive linear maps and criterion for separability

Following Ref. [8] we call a state  $\varrho$  acting on  $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$  *separable* if it admits decomposition into a convex combination of products of states describing local systems as in (1). The most celebrated criterion allowing to distinguish separable states from the entangled ones is the criterion based on positive maps [5, 6] (see also Ref. [22]).

We say that a linear map  $\Lambda : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  is *positive* if when acting on a positive element from  $M_m(\mathbb{C})$ , it returns positive element from  $M_n(\mathbb{C})$ . If, moreover, the extended map  $I_d \otimes \Lambda : M_d(\mathbb{C}) \otimes M_m(\mathbb{C}) \rightarrow M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$ , where  $I_d$  denotes identity map, is positive for any  $d$ , we call  $\Lambda$  *completely positive* (CP). A linear map  $\Lambda$  that do not change trace of the object on which it acts, i.e.,  $\text{Tr}[\Lambda(X)] = \text{Tr} X$  is satisfied for any  $X \in M_m(\mathbb{C})$ , is called *trace-preserving* (TP). Completely positive maps which are additionally TP are called *quantum channels*. Let us mention that there is a simple criterion allowing to judge if a given map is CP [23]. Namely,  $\Lambda : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  is CP iff  $(I \otimes \Lambda)(P_m^+) \geq 0$ , where  $P_m^+$  denotes a projector onto the maximally entangled state

$$|\psi_m^+\rangle = \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} |i\rangle|i\rangle. \quad (2)$$

A well-known example of the positive, but not completely positive map is the transposition map  $T : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ . It is clear that when applied to a positive operator  $X$  it returns also a positive operator  $X^T$ . On the other hand, local application of the transposition map (the so-called partial transposition) to one of the subsystems of the maximally entangled states gives operator  $(I \otimes T)(P_m^+) \equiv (P_m^+)^T$  which is not

positive. In what follows, a density matrix  $\varrho$  such that  $(I \otimes T)(\varrho) \geq 0$  ( $(I \otimes T)(\varrho) \not\geq 0$ ) will be called PPT (NPT).

A good example of a completely positive trace-preserving map is the depolarizing channel  $D[p] : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  defined as

$$D[p](\varrho) = (1 - p) \operatorname{Tr}(\varrho) \frac{\mathbb{1}_m}{m} + p\varrho \quad (0 \leq p \leq 1). \quad (3)$$

When applied to a state  $\varrho \in M_m(\mathbb{C})$  it returns  $\varrho$  with probability  $p$  and the maximally mixed state  $\mathbb{1}_m/m$  with probability  $1 - p$ . This map is CP because its partial application to  $P_m^+$  gives the known isotropic states [23]:

$$\varrho_{\text{iso}}(p) = \frac{1 - p}{m^2} \mathbb{1}_{m^2} + pP_m^+. \quad (4)$$

For brevity, we denote by  $D$  the completely depolarizing channel  $D[1](\varrho) = \operatorname{Tr}(\varrho) \mathbb{1}_m/m$ .

The importance of positive maps in entanglement detection comes from the fact that when acting locally on separable states, they leave them in a separable form, and thus  $(I \otimes \Lambda)(\varrho_{\text{sep}}) \geq 0$  for any separable  $\varrho_{\text{sep}}$ , and any positive map  $\Lambda$ . More importantly, it was proven in Ref. [6] that a given state  $\varrho$  acting on  $\mathcal{H}_{m,n}$  is separable if and only if the operator  $(I \otimes \Lambda)(\varrho)$  is positive for any positive map  $\Lambda : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ . This means that for any entangled state there exist a positive map detecting it. Since, however, the structure of positive maps is still not well understood, the sufficient part of this criterion is not easy-to-apply. Nevertheless, it gives a very strong *necessary criterion for separability*: given a state  $\varrho$  and a map  $\Lambda$ , check if the operator  $(I \otimes \Lambda)(\varrho)$  has negative eigenvalues, and if it does, then  $\varrho$  is entangled.

Within the set of positive maps let us distinguish the subset of *decomposable maps* as the ones that admit the decomposition  $\Lambda = \Lambda_1^{\text{CP}} + \Lambda_2^{\text{CP}} \circ T$  with  $\Lambda_i^{\text{CP}} : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  denoting the completely positive maps, and  $T$  the transposition map. All the remaining maps are called *indecomposable*. Interestingly, since any positive map  $\Lambda : M_n(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  with  $n = 2, 3$  is decomposable [25, 26], in the case of qubit-qubit and qubit-qutrit systems we have a unique test unambiguously detecting all entangled states. More precisely, all the entangled states acting on  $\mathbb{C}^2 \otimes \mathbb{C}^m$  with  $m = 2, 3$  are detected by the transposition map.

Finally, among all the CP maps let us distinguish those which, when partially applied to any bipartite state, give in return separable states only. Such maps are called *entanglement breaking* (EB), and if they are additionally TP, they are called *entanglement breaking channels* (EBCs). These maps were investigated in Ref. [18] (see also Ref. [27] and [49] for the characterization of EB channels in continuous variables systems), and the sufficient and necessary criterion allowing to check if a given map is EB was worked out. Precisely, given CP map  $\Lambda : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  it is EB iff the positive operator  $(I \otimes \Lambda)(P_m^+)$  can be written in a separable form (1). Also, it was shown in Ref. [18] that any entanglement breaking channel can be represented in the Holevo form [28], i.e., its action can be written as

$$\Lambda(\varrho) = \sum_k \operatorname{Tr}(F_k \varrho) \varrho_k, \quad (5)$$

where the operators  $F_k \in M_m(\mathbb{C})$  acting on represent some generalized quantum measurement, i.e., they are positive and sum up to  $\mathbb{1}_m$ , and  $\varrho_k$  stand for some density matrices acting on  $\mathcal{H}_m$ .

## 2.2. Entanglement witnesses and their optimality

From experimental point of view it is important to notice that the above separability criterion can be reformulated in terms of mean values of some physical observables [6]. To be more precise let us recall that the Choi-Jamiołkowski (C-J) isomorphism [29, 30] assigns to every positive map  $\Lambda : M_n(\mathbb{C}) \rightarrow M_{n'}(\mathbb{C})$ , a Hermitian operator

$$W_\Lambda = (I \otimes \Lambda^\dagger)(P_{n'}^+), \quad (6)$$

acting on  $M_{n'}(\mathbb{C}) \otimes M_n(\mathbb{C})$  with  $\Lambda^\dagger : M_{n'}(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  denoting the dual map of  $\Lambda$ .

Since mean values of operators given by Eq. (6) on separable states are always nonnegative, and, on the other hand, for any  $W$  there exists an entangled state  $\rho$  for which  $\langle W \rangle_\rho < 0$ , we call these operators *entanglement witnesses* [31].

Via the C-J isomorphism, the division of positive maps to decomposable and indecomposable can be translated to EWs. More specifically, the decomposable positive maps give *decomposable entanglement witnesses* which, as it follows directly from Eq. (6), take the general form  $W = P + Q^\Gamma$  with  $P, Q \geq 0$ . If a given witness cannot be written in this form we call it *indecomposable*. They can be written as  $W_{\text{in}} = P + Q^\Gamma - \epsilon \mathbb{1}$  with  $0 < \epsilon \leq \inf_{|e, f\rangle} \langle e, f | P + Q^\Gamma | e, f \rangle$  and as previously  $P, Q \geq 0$  and there exists some PPT edge state§ [32] such that its kernel is equal to the support of  $P$ , while kernel of its partial transposition is equal to the support of  $Q$  [33] (see also Ref. [34]).

Of all the EWs the ones which merit distinction are the *optimal* ones. Precisely, following Ref. [35], we call  $W$  *finer* than  $W'$  if the set of states detected by  $W$  (denoted  $D_W$ ) contains the set of states detected by  $W'$ , i.e.,  $D_{W'} \subseteq D_W$ . The witness  $W$  is called optimal if there does not exist a witness finer than  $W$ . Let us briefly remind some relevant facts about optimal witnesses obtained in Ref. [35]. The above definition means that  $W$  is optimal if and only if  $(1 + \epsilon)W - \epsilon P$  is no longer an EW for any  $\epsilon > 0$  and  $P \geq 0$ . Then, if the set  $\mathcal{P}_W = \{|e, f\rangle \langle e, f | W | e, f \rangle = 0\}$  spans the Hilbert space on which  $W$  acts then it is optimal. These facts immediately imply that decomposable witnesses are only those which take the form  $W = Q^\Gamma$  with positive  $Q$  which does not contain any product vectors in its support.

Restricting to indecomposable witnesses only, the above properties have to be reformulated. An operator  $W$  is indecomposable optimal entanglement witness iff for any  $\epsilon > 0$  and  $P, Q \geq 0$ , the operator  $(1 + \epsilon)W - \epsilon(P + Q^\Gamma)$  is not an entanglement witness (one cannot subtract a decomposable operator). Moreover, if both  $\mathcal{P}_W$  and  $\mathcal{P}_{W^\Gamma}$  span the whole Hilbert space, then  $W$  is indecomposable optimal entanglement witness.

One could conjecture that the above conditions for optimality of witnesses is also sufficient. There exists, however, an example of an indecomposable map provided by Choi [36] (see Sec. 3.2) such that its  $\mathcal{P}_W$  does not span the corresponding Hilbert space, but it leads to a optimal entanglement witness. The latter follows from the fact that the Choi map was proven to be extremal in the set of positive maps [37].

## 2.3. Structural physical approximation and the conjecture

The C-J isomorphism tells us that both formulations of the above separability criterion are equivalent. Nevertheless, a given positive map detects more states than

§ We call  $\delta \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$  a *PPT edge state* if it is entangled PPT state and for any product vector  $|e, f\rangle \in \mathcal{H}_{mn}$  and  $\epsilon > 0$  the operator  $\delta - \epsilon |e, f\rangle \langle e, f|$  either is not positive or it has nonpositive partial transposition.

the corresponding entanglement witness. The problem with the formulation of the criterion in terms of positive maps lies in the fact that generically they do not represent any physical process. Consequently, they cannot be implemented experimentally. Since, they are more efficient in entanglement detection, it has been an interesting question how to make positive maps experimentally implementable.

One way to solve this problem employs the *structural physical approximation* (SPA) [9, 10, 11]. Roughly speaking, the idea behind the SPA comes from the fact that any positive, but not completely positive map  $\Lambda$  (actually it does not have to be even positive), when mixed with sufficiently large amount of the completely depolarizing channel  $D$ , will result in CP map. Precisely, the idea is to take the following mixture

$$\tilde{\Lambda}[p] = (1-p)D + p\Lambda \quad (0 \leq p \leq 1). \quad (7)$$

From the C-J isomorphism it is clear that there exists a nontrivial parameter range  $0 \leq p \leq p_*$  with  $p_* > 0$  for which the map  $\tilde{\Lambda}[p]$  is completely positive (represents quantum channel if the initial map  $\Lambda$  is TP), and thus, in principle, it can represent some physical process. The least noisy completely positive map from the class  $\tilde{\Lambda}[p]$  ( $0 \leq p \leq p_*$ ), i.e.,  $\tilde{\Lambda}[p_*]$  is called structural physical approximation of  $\Lambda$ .

In the language of entanglement witnesses SPA is a positive operator obtained by mixing a given witness  $W$  from  $M_{m,n}$  with maximally mixed state  $\mathbb{1}_m \otimes \mathbb{1}_n / mn$ , i.e.,

$$\tilde{W}(p) = (1-p) \frac{\mathbb{1}_m \otimes \mathbb{1}_n}{mn} + pW \quad (0 \leq p \leq 1). \quad (8)$$

Since the maximally mixed state is represented by a full rank positive matrix it is clear that there exists a threshold value for  $p$  (denoted by  $p_*$ ) for which  $\tilde{W}(p)$  becomes positive. Simple calculations show that it is given by  $p_* = 1/(1+mn\lambda)$ , where  $-\lambda < 0$  is the smallest eigenvalue of  $W$ .

Having all the necessary notions and tools we can recall the conjecture formulated in Ref. [19].

**Conjecture.** *Let  $\Lambda$  be an optimal (trace-preserving) positive map. Then, its SPA is entanglement breaking map (channel).*

The difficulty of proving or disproving the conjecture comes from whether the density matrices resulting from an application of SPA to EWs are separable, and the separability problem still remains unsolved.

In what follows, basing on what was said in Sec. 2.2, we prove two facts relating the notions of SPA and optimality of EWs.

**Fact 1.** *Let  $W_1$  and  $W_2$  be two entanglement witnesses. If  $W_1$  is finer than  $W_2$  then  $p_*^{(1)} \leq p_*^{(2)}$ , where  $p_*^{(1)}$  and  $p_*^{(2)}$  are threshold values in (8) of  $W_1$  and  $W_2$ , respectively. In this sense optimal witnesses have smallest  $p_*$ .*

**Proof.** Let  $D_{W_1}$  and  $D_{W_2}$  be the sets of states detected by  $W_1$  and  $W_2$ , respectively. Since  $W_1$  is finer than  $W_2$ ,  $D_{W_2} \subseteq D_{W_1}$ . Let also  $-\lambda_{1(2)} < 0$  denote the smallest eigenvalue of  $W_1$  ( $W_2$ ). It is clear that

$$-\lambda_{1,2} = \min_{\rho \in D_{W_{1,2}}} \text{Tr}(W_{1,2}\rho). \quad (9)$$

On the other hand, we already know that  $p_*^{(1,2)} = 1/(1 + mn\lambda_{1,2})$ . Then, in order to prove that  $p_*^{(1)} \leq p_*^{(2)}$ , it suffices to apply the following estimation

$$\begin{aligned} -\lambda_1 &= \min_{\rho \in D_{W_1}} \text{Tr}(W_1 \rho) \\ &\leq \min_{\rho \in D_{W_2}} \text{Tr}(W_1 \rho) \\ &\leq \min_{\rho \in D_{W_2}} \text{Tr}(W_2 \rho) = -\lambda_2, \end{aligned} \quad (10)$$

where the second inequality is a consequence of the fact that  $W_1$  is finer than  $W_2$  and therefore  $\text{Tr}(W_1 \rho) \leq \text{Tr}(W_2 \rho)$  for all  $\rho \in D_{W_2}$ . ■

**Fact 2.** Suppose that  $W_1$  is finer than  $W_2$ . Then, the SPA of  $W_1$ ,  $\widetilde{W}_1$ , cannot be detected by  $W_2$ .

**Proof.** Since  $W_1$  is finer than  $W_2$ , one can express the latter as  $W_2 = (1 - \epsilon)W_1 + \epsilon P$  for some  $\epsilon > 0$  and  $P \geq 0$ . Since

$$\text{Tr}(\widetilde{W}_1 W_1) = p_*^{(1)} \text{Tr} W_1^2 + [(1 - p_*^{(1)})/m^2] \text{Tr} W_1 \geq 0, \quad (11)$$

and  $\widetilde{W}_1 \geq 0$ , it is straightforward to see

$$\text{Tr}(\widetilde{W}_1 W_2) = (1 - \epsilon) \text{Tr}(\widetilde{W}_1 W_1) + \epsilon \text{Tr}(\widetilde{W}_1 P) \geq 0, \quad (12)$$

which completes the proof. ■

This fact shows that any non-optimal EW  $W_2$  that can be optimized to an optimal EW  $W_1$  cannot detect the SPA  $W_1$ . Hence, provided  $\widetilde{W}_1$  is entangled, the EW detecting it cannot be the one that can be optimized to  $W_1$ . This, somehow, reflects the difficulty of proving or disproving the conjecture.

### 3. Extending the set of witnesses which support the conjecture

The main aim of this section is to provide new examples of optimal entanglement witnesses which obey the above conjecture.

#### 3.1. All decomposable witnesses with $Q$ of rank one

Let us start from the decomposable witnesses being partial transpositions of pure entangled states.

**Theorem 1.** Let  $W = |\psi\rangle\langle\psi|^\Gamma$  be a witness with entangled  $|\psi\rangle \in \mathcal{H}_m$ . Then its SPA is entanglement breaking channel.

**Proof.** First we write  $|\psi\rangle$  in its Schmidt decomposition:

$$|\psi\rangle = \sum_{i=0}^{r-1} \mu_i |ii\rangle, \quad (13)$$

where  $r > 1$ , and  $\mu_i \geq 0$  denote the Schmidt rank and Schmidt coefficients of  $|\psi\rangle$ , respectively. Also, without any loss of generality, we can set Schmidt local bases to the computational ones. Following Eq. (8), the structural physical approximation of  $W$  reads

$$\widetilde{W}(p) = (1 - p) \frac{\mathbb{1}_m \otimes \mathbb{1}_m}{m^2} + p |\psi\rangle\langle\psi|^\Gamma. \quad (14)$$

It is clear that the eigenvalues of  $|\psi\rangle\langle\psi|^\Gamma$  are  $\mu_i^2$  ( $i = 1, \dots, r$ ) and  $\pm\mu_i\mu_j$  ( $i \neq j = 1, \dots, r$ ). Therefore, denoting by  $\theta$  the largest of the eigenvalues  $\mu_i\mu_j$  ( $i \neq j$ ), we see that  $\widetilde{W}(p) \geq 0$  iff  $p \leq 1/(d^2\theta + 1)$ . Now, our aim is to prove that for  $p_* = 1/(d^2\theta + 1)$ , the state  $\widetilde{W}(p_*)$  is separable. For this purpose, let us consider the following class of states

$$\rho(p) \equiv [\widetilde{W}(p)]^\Gamma = (1-p) \frac{\mathbb{1}_m \otimes \mathbb{1}_m}{m^2} + p|\psi\rangle\langle\psi|. \quad (15)$$

If the state  $\rho(p)$  is separable for  $p = p_*$  then obviously  $[\rho(p_*)]^\Gamma = \widetilde{W}(p_*)$  also is. For the threshold value  $p = p_*$ , the above reduces to  $\rho(p_*) = [1/(d^2\theta + 1)] (\theta \mathbb{1}_m \otimes \mathbb{1}_m + |\psi\rangle\langle\psi|)$ , which after simple movements can be rewritten as

$$\begin{aligned} \rho(p_*) = \frac{1}{d^2\theta + 1} & \left( \theta \sum_{i=0}^{r-1} |ii\rangle\langle ii| + \sum_{\substack{i,j=0 \\ i \neq j}}^{r-1} (\theta - \mu_i\mu_j) |ij\rangle\langle ij| + \theta \sum_{i,j=r}^{m-1} |ij\rangle\langle ij| \right. \\ & \left. + \sum_{\substack{i,j=0 \\ i \neq j}}^{r-1} \mu_i\mu_j |ij\rangle\langle ij| + \sum_{i,j=0}^{r-1} \mu_i\mu_j |ii\rangle\langle jj| \right). \end{aligned} \quad (16)$$

Due to the fact that  $\theta \geq \mu_i\mu_j$  for any  $i < j$  ( $i, j = 0, \dots, r-1$ ), one sees that first three terms represent separable states. Consequently, what remains is to prove the separability of the last two terms. It is clear that they give a positive matrix, and thus represent a (unnormalized) quantum state. After normalization we can rewrite them as

$$\bar{\rho}_\mu = \frac{1}{N} \left( D_\mu \otimes D_\mu + \sum_{i \neq j} \mu_i\mu_j |ii\rangle\langle jj| \right), \quad (17)$$

where by  $D_\mu$  we denote a matrix diagonal in the standard basis with eigenvalues  $\mu_i$  ( $i = 0, \dots, r-1$ ), while  $N = (\text{Tr} D_\mu)^2$ . Separability of this state for any configuration of  $\mu$  was proven in Ref. [45]. Consequently, we see that  $\widetilde{W}(p_*)$  can be written as a convex combination of separable states and therefore is itself separable. ■

Let us remark that, since every witness  $W = |\psi\rangle\langle\psi|^\Gamma$  with entangled  $|\psi\rangle$  is extremal in the convex set of entanglement witnesses, it is also optimal. Also, notice that the above theorem can be extended easily to the case of unequal dimension on both sites.

### 3.2. A class of indecomposable positive maps

In Ref. [19] the conjecture was proven for the Choi map [36]. Here, we show that this analysis can be extended to one of the generalizations of Choi map analyzed in a series of papers [38, 39, 40]. Their action on any  $X \in M_m(\mathbb{C})$  can be expressed as

$$\tau_{m,k}(X) = (m-k)\varepsilon(X) + \sum_{i=1}^k \varepsilon(S^i X S^{i\dagger}) - X, \quad (18)$$

where  $k = 1, \dots, m-1$  and the completely positive map  $\varepsilon$  is given by  $\varepsilon(X) = \sum_{i=0}^{m-1} \langle i|X|i\rangle |i\rangle\langle i|$ . Note that  $k = m-1$  the map  $\tau_{m,m-1}$  is just the reduction map  $R_-(X) = \text{Tr}(X) \mathbb{1}_m - X$  [23, 24]. For the remaining cases,  $k = 1, \dots, m-2$ , the map



is indecomposable, and in particular  $\tau_{3,1}$  is the Choi map. Eventually,  $S$  is a shift operator acting as  $S|i\rangle = |i+1\rangle \pmod{m}$ .

Now, let us introduce the normalized version of the above map, that is  $\mathcal{T}_{m,k} = [1/(m-1)]\tau_{m,k}$ , and notice that it is TP and unital $\|$ . Acting with the extended map  $I \otimes \mathcal{T}_{m,k}$  on the maximally entangled state, one obtains

$$W_{m,k} = \frac{1}{m(m-1)} \left[ (m-k)\varrho_0 + \sum_{i=1}^k \varrho_i - mP_m^+ \right] \quad (19)$$

with  $\varrho_i$  ( $i = 0, \dots, k$ ) denoting (unnormalized) separable density matrices of the form

$$\varrho_i = \sum_{l=0}^{m-1} |l\rangle\langle l| \otimes |l+i\rangle\langle l+i|, \quad (20)$$

where the summation is modulo  $m$ . By mixing  $W_{m,k}$  with the maximally mixed state, we arrive at

$$\widetilde{W}_{m,k}(p) = \frac{1-p}{m^2} \mathbb{1}_{m^2} + pW_{m,k}, \quad (21)$$

which, due to the fact that the smallest eigenvalue of  $W_{m,k}$  is  $-k/[m(m-1)]$ , is positive iff  $p \leq [(m-1)/(m(k+1)-1)]$ . Now, we need to show that for  $p_* = (m-1)/[m(k+1)-1]$ , the matrix  $\widetilde{W}_{m,k}(p_*)$  represents a separable state. Let us then rewrite it as

$$\widetilde{W}_{m,k}(p_*) = \frac{1}{N} \left[ (m-k)\varrho_0 + k\mathbb{1}_{m^2} - mP_m^+ + \sum_{i=1}^k \varrho_i \right], \quad (22)$$

where  $N = m[m(k+1)-1]$ . As the last term in the square brackets in the above is already separable (cf. (20)), our aim is to prove the separability of the first three terms. For this purpose, we can follow the reasoning applied to the Choi map in Ref. [19]. More precisely, we can rewrite these terms as

$$(m-k)\varrho_0 + k\mathbb{1}_{m^2} - mP_m^+ = \sum_{\substack{i,j=0 \\ i < j}}^{m-1} \sigma_{ij} + (k-1)(\mathbb{1}_{m^2} - \varrho_0), \quad (23)$$

where  $\sigma_{ij} = |ii\rangle\langle ii| + |jj\rangle\langle jj| + |ij\rangle\langle ij| + |ji\rangle\langle ji| - |ii\rangle\langle jj| - |jj\rangle\langle ii|$  stand for two-qubit matrices embedded in  $M_{m^2}(\mathbb{C})$ . One easily checks that they are PPT and thus separable. The second term, being diagonal in the standard basis in  $\mathcal{H}_m$ , is also separable. Consequently,  $\widetilde{W}_{m,k}(p_*)$  represents a separable state for all possible values of  $m$  and  $k$ , meaning the SPAs of  $\mathcal{T}_{m,k}$  are EB channels.

Concluding, let us comment on the optimality of the witnesses  $W_{m,k}$ . In Ref. [19] it was shown that in the case of the Choi map,  $\tau_{3,1}$ , the set  $\mathcal{P}_{W_{3,1}}$  do not span the corresponding Hilbert space  $\mathbb{C}^3 \otimes \mathbb{C}^3$ . Nevertheless, since  $\tau_{3,1}$  is extremal, it has to be optimal as one can subtract neither a decomposable witness nor a positive operator. The reasoning from Ref. [19] can be extended to all maps  $\tau_{m,k}$  ( $k = 1, \dots, m-1$ ). Specifically, one checks directly that the product vectors from  $\mathcal{P}_{W_{m,k}}$  can be written as

$$\sum_{k=0}^{m-1} e^{i\phi_k} |k\rangle \otimes \sum_{k=0}^{m-1} e^{-i\phi_k} |k\rangle \quad (\phi_0 = 0). \quad (24)$$

Clearly, they span  $m^2 - m + 1$  dimensional subspace of  $\mathcal{H}_m$ , and therefore the problem of optimality of these witnesses (except for  $W_{3,1}$ ) is open.

$\|$  We call the map  $\Lambda : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  unital if  $\Lambda(\mathbb{1}_m) = \mathbb{1}_m$ .

### 3.3. The conjecture and the isotropic states

Very recently a simple, but useful fact relating the conjecture, and the isotropic states (4) was proven [41, 20]. It can be stated as follows.

**Theorem 2** [41, 20]. *Let  $\Lambda : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  be a unital map that detects all the entangled isotropic states  $\varrho_{\text{iso}}(p)$  (cf. (4)). Then SPA of  $\Lambda$  is an entanglement breaking map.*

The aim of this section is two-fold. On the one hand, we determine the class of decomposable positive maps which satisfy assumptions of this theorem and therefore obey the conjecture. On the other hand, we provide a simple generalization of this theorem to the class of maps containing, among others, unital maps.

In order to look for the witnesses satisfying theorem 2, let us rephrase it slightly. Denoting by  $\Lambda$  some unital map, the condition of detecting all the entangled isotropic states by this map is equivalent to the condition that there exists a pure entangled state  $|\psi\rangle \in \mathcal{H}_m$  such that

$$\text{Tr}[(I \otimes \Lambda^\dagger)(P_\psi)\varrho_{\text{iso}}(\frac{1}{d+1})] = \text{Tr}[(A \otimes \mathbb{1}_m)\omega(A^\dagger \otimes \mathbb{1}_m)\varrho_{\text{iso}}(\frac{1}{d+1})] = 0. \quad (25)$$

Here  $P_\psi = |\psi\rangle\langle\psi|$ ,  $\omega = (I \otimes \Lambda^\dagger)(P_m^+)$  denotes the witness corresponding to  $\Lambda$  via the C-J isomorphism, and  $A$  stands for the matrix such that  $|\psi\rangle = A \otimes \mathbb{1}_m |\psi_m^+\rangle$ . Notice that since  $\Lambda$  is unital, its dual map  $\Lambda^\dagger$  is TP. It is now clear that our aim is to find the general form of decomposable witnesses  $\omega$  satisfying the above condition.

Since, by assumption,  $\Lambda$  is decomposable,  $\omega = P + Q^\Gamma$  with  $P, Q \geq 0$  (see Sec. 2.2). Also, we know that  $\Lambda$  is unital if and only if the corresponding EW  $\omega$  has a maximally mixed first subsystem ( $\text{Tr}_B \omega = \mathbb{1}_m/m$ ) meaning that  $P$  and  $Q$  have to obey  $\text{Tr}_B(P + Q) = \mathbb{1}_m/m$ .

Denoting  $P_A = (A \otimes \mathbb{1}_m)P(A^\dagger \otimes \mathbb{1}_m)$  (the same for  $Q$ ), we can now rewrite the condition (25) as

$$\text{Tr}[P_A \varrho_{\text{iso}}(\frac{1}{d+1})] + \text{Tr}[Q_A \varrho_{\text{iso}}^\Gamma(\frac{1}{d+1})] = 0. \quad (26)$$

We know that  $\varrho_{\text{iso}}(\frac{1}{d+1})$  is separable and therefore PPT. This together with positivity of  $P_A$  and  $Q_A$  allows us to infer that both terms in the above equation have to be zero, i.e.,

$$\text{Tr}[P_A \varrho_{\text{iso}}(\frac{1}{d+1})] = 0, \quad \text{Tr}[Q_A \varrho_{\text{iso}}^\Gamma(\frac{1}{d+1})] = 0. \quad (27)$$

From the explicit form of  $\varrho_{\text{iso}}(\frac{1}{d+1}) = [1/d(d+1)](\mathbb{1}_{m^2} + dP_+^{(m)})$  one sees that it is of full rank and hence the first condition in Eq. (27) yields  $P_A = 0$ . On the other hand, one checks that  $\varrho_{\text{iso}}^\Gamma(\frac{1}{d+1}) = [2/d(d+1)]P_s$  with  $P_s$  denoting the projector onto the symmetric subspace of  $\mathcal{H}_m$  (henceforward denoted  $\mathcal{S}(\mathcal{H}_m)$  or shortly  $\mathcal{S}_m$ ). This, when applied to the second condition in Eq. (27), implies that  $Q_A$  has to be supported $\P$  on the antisymmetric subspace of  $\mathcal{H}_m$  (henceforward denoted  $\mathcal{A}(\mathcal{H}_m)$  or shortly  $\mathcal{A}_m$ ).

Let us remind that the antisymmetric subspace of  $\mathcal{H}_m$  is spanned by the vectors  $|\varphi_{ij}^-\rangle = (1/\sqrt{2})(|ij\rangle - |ji\rangle)$  ( $i, j = 0, \dots, m-1$ ,  $i < j$ ). Consequently, the general form of  $Q_A$  is given by

$$Q_A = \sum_{i < j} \sum_{k < l} \alpha_{ij,kl} |\varphi_{ij}^-\rangle\langle\varphi_{kl}^-|, \quad (28)$$

$\P$  By saying that a positive operator  $Q$  is supported on a subspace  $V$  we mean that its eigenstates corresponding to nonzero eigenvalues belong to  $V$ .

where  $\alpha$ s must obey  $\alpha_{ij,kl} = \alpha_{kl,ij}^*$  (to guarantee Hermiticity) and corresponding positivity conditions.

The general conclusion following the above analysis is that if we have a decomposable witness  $\omega = P + Q^\Gamma$  with  $P, Q \geq 0$  satisfying the condition  $\text{Tr}_B(P + Q) = \mathbb{1}_m/m$  and there exists a matrix  $A$  such that  $P_A = 0$  and  $Q_A$  supported on  $\mathcal{A}_m$ , then the corresponding map  $\Lambda$  is unital and detects all the isotropic states implying that its SPA is entanglement breaking.

Still more can be said about witnesses that satisfy assumption of the theorem 2. First, let us notice that in general  $r(A) \geq 2$ . This follows from the fact that rank-one  $A$  lead to  $Q_A$  of the separable form, and such  $Q_A$  cannot be supported on  $\mathcal{A}_m$ . On the other hand, for full rank  $A$  the identity  $P_A = 0$  implies  $P = 0$ . Moreover, in such case the relevant  $Q$ s are given by  $(A^{-1} \otimes \mathbb{1}_m) \tilde{Q} (A^{-1} \otimes \mathbb{1}_m)^\dagger$  with  $\tilde{Q}$  of the form (28) with  $\alpha$ s satisfying the following system of equations

$$\sum_{\substack{i=0 \\ q < i}}^{m-1} \alpha_{pi,qi} + \sum_{\substack{i=0 \\ i < p}}^{m-1} \alpha_{ip,iq} - \sum_{\substack{i=0 \\ p < i < q}}^{m-1} \alpha_{pi,iq} = \frac{2}{m} \langle p | A A^\dagger | q \rangle$$

$$(p, q = 0, \dots, m-1, p \leq q). \quad (29)$$

In the particular case  $A = \mathbb{1}_m$ , the above analysis tells us that all witnesses  $W = Q^\Gamma$  with  $Q$  supported on  $\mathcal{A}_m$ , and with maximally mixed first subsystem, satisfy the conjecture. This particular result is a generalization of the analysis done for  $3 \otimes 3$  systems in Ref. [19].

Let us now discuss the case of full rank  $A$ . In the case of  $m = 2$ , the antisymmetric subspace of two-qubit space is one-dimensional and spanned by  $|\varphi_{01}^- \rangle$ . To find  $Q$  we need to solve the equation  $Q_A = \alpha P_{01}^-$  with  $P_{01}^-$  denoting projector onto  $|\varphi_{01}^- \rangle$ . Since  $\text{Tr}_B P_{01}^- = \mathbb{1}_2/2$ , this equation implies that  $AA^\dagger = \alpha \mathbb{1}_2$ . We know, however, that  $A$  corresponds to normalized pure state and therefore  $\text{Tr}(AA^\dagger) = 2$ , which implies that  $\alpha = 1$  and consequently  $A$  can only be a unitary matrix. Thus, in the two-qubit case only witnesses being local unitary rotations of  $P_{01}^-$  are "detected" by theorem 2.

In the case of  $m = 3$ ,  $\mathcal{A}_3$  is three-dimensional and spanned by the vectors  $|\varphi_{01}^- \rangle$ ,  $|\varphi_{02}^- \rangle$ , and  $|\varphi_{12}^- \rangle$ . Every  $Q$  defined on  $\mathcal{A}_3$  is characterized by nine real parameters. On the other hand, for any nonsingular  $A$ , (29) give us exactly nine equations for  $\alpha$ s, meaning that generically for any such  $A$  there exists a unique solution to (29), and thus a unique witness  $\omega$  that satisfies the conjecture. In the particular case of  $A = \mathbb{1}_m$  this witness is given by  $W = (1/3)(P_{01}^- + P_{02}^- + P_{12}^-)^\Gamma$ .

Generically (except for the case  $m = 2$ ) the system (29) gives us  $m$  equations for  $m(m-1)/2$  real parameters  $\alpha_{ij,ij}$  and  $m(m-1)/2$  equations for  $m(m-1)[m(m-1)-2]/8$  for complex off-diagonal  $\alpha_{ij,kl}$ . Thus, we have in total  $m^2$  equations for  $m^2(m-1)^2/4$  real parameters. Consequently, for any nonsingular  $A$  corresponding to a pure normalized state  $|\psi\rangle$ , we have a  $[m^2(m+1)(m-3)/4]$ -parameter class (without taking into account positivity conditions) of witnesses satisfying the conjecture. It should be also emphasized that all such witnesses are optimal. This follows from the fact that since the support of  $Q_A$  belongs to  $\mathcal{A}_m$ , its kernel contains all the vectors  $(|0\rangle + \alpha|1\rangle + \dots + \alpha^{m-1}|m-1\rangle)^{\otimes 2}$  with  $\alpha \in \mathbb{C} \cup \{\infty\}$ . Then, one knows that the partial conjugated vectors  $(|0\rangle + \alpha|1\rangle + \dots + \alpha^{m-1}|m-1\rangle) \otimes (|0\rangle + \alpha^*|1\rangle + \dots + (\alpha^*)^{m-1}|m-1\rangle)$  span the whole Hilbert space  $\mathcal{H}_m$ . Since  $Q$  is related to  $Q_A$  via a local nonsingular transformation,  $Q^\Gamma$  has mean values on product vectors that also span  $\mathcal{H}_m$  and, according to what was said in Sec. 2.2, it is optimal.

In the remaining cases with respect to the rank of  $A$ , i.e.,  $2 \leq r(A) \leq m-1$ , characterization of witnesses which are detected by theorem 2 is a nontrivial task. Still, however, the above analysis allows us to provide examples of entanglement witnesses  $W = P + Q^\Gamma$  obeying the conjecture with are certainly not optimal. For instance let  $A = \sum_{i=0}^{k-1} |i\rangle\langle i|$  ( $r(A) = k < m$ ) and  $P = (1/m) \sum_{i=k}^{m-1} |ii\rangle\langle ii|$ , and  $Q = [2/m(k-1)] \sum_{0 \leq i < j \leq k-1} P_{ij}^-$ . Then, one easily finds that  $P_A = 0$  and  $Q_A = Q$ . Finally,  $\text{Tr}_B W = \text{Tr}_B(P + Q) = \mathbb{1}_m/m$  and therefore  $W$  and the corresponding map obey the conjecture, nevertheless,  $W$  is certainly not optimal.

Interestingly, using theorem 1, theorem 2 can be generalized to a class of maps which contains unital maps as a special case. More precisely, we have the following statement.

**Theorem 3.** *Let  $\Lambda : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  be a positive map of the form  $\Lambda = \Lambda_{\text{un}} \circ \Lambda_A$ , where  $\Lambda_A(\cdot) = A(\cdot)A^\dagger$  and  $\Lambda_{\text{un}}$  is a unital map detecting all entangled states in the class (15) with  $|\psi\rangle = \mathbb{1}_m \otimes A|\psi_m^+\rangle$ . Then SPA of  $\Lambda$  is entanglement breaking.*

**Proof.** The proof goes along the same lines as the one given in Ref. [20]. The structural physical approximation of  $\Lambda$  is given by  $\tilde{\Lambda}[p] = (1-p)D + p\Lambda$ . In order to check for which values of  $p$  it is CP, let us notice that the partial application of  $\tilde{\Lambda}[p]$  to  $P_m^+$  can be expressed as

$$\begin{aligned} (I \otimes \tilde{\Lambda}[p])(P_m^+) &= \frac{1-p}{m^2} \mathbb{1}_m \otimes \mathbb{1}_m + p(I \otimes \Lambda)(P_m^+) \\ &= (I \otimes \Lambda_{\text{un}})(\rho(p)), \end{aligned} \quad (30)$$

where to get the second equality we utilized the fact that  $\Lambda = \Lambda_{\text{un}} \otimes \Lambda_A$  and  $\mathbb{1}_m \otimes A|\psi_m^+\rangle = |\psi\rangle$ . Now, we know that  $\rho(p)$  is separable for  $p \leq 1/(d^2\theta + 1)$  and therefore  $(I \otimes \tilde{\Lambda}_p)(P_m^+)$  is positive and separable for these values of  $p$ . On the other hand, by assumption,  $\Lambda_{\text{un}}$  detects all the entangled states  $\rho(p)$  meaning that  $\tilde{\Lambda}[p]$  is not CP for  $p > 1/(d^2\theta + 1)$ . Consequently, SPA of  $\Lambda$  is entanglement breaking. ■

A particular example of a map satisfying assumptions of the above fact is  $\Lambda = T \circ \Lambda_A$  with  $A$  being any matrix. It is clear that  $T$  is unital and detects all the entangled states  $\rho(p)$ . On the other hand, one sees that maps of the form  $T \circ \Lambda_A$  do not satisfy assumptions of theorem 2.

#### 4. Extending the conjecture

It is interesting to ask whether other than  $D$  EBCs when used to construct SPA (according to (7)) can also lead to EBCs. Or, in other words, if all separable states which when put in Eq. (8) instead of the maximally mixed state, will also give separable states. Certainly, these can only be the full rank separable states as otherwise the negative eigenvalues of some witness  $W$  would not be "covered".

It is the purpose of this section to explore the above questions. We give an example of some other than  $D$  entanglement breaking channel that when used for construction of SPAs does not necessarily lead to an entanglement breaking map. On the other hand, it is shown that for any positive map (EW) there exists an EB channel (separable state) such that the SPA constructed in this way is also EB (separable). Finally, we comment on the possibility of extension of the conjecture to the case of Gaussian states.

## 4.1. SPAs using other entanglement breaking channels

Let  $\Phi$  denote some entanglement breaking channel such that the separable state  $\sigma_{\text{sep}} = (I \otimes \Phi)(P_m^+)$  is of full rank. Then let us consider the following mixture

$$\tilde{\Lambda}[p] = (1-p)\Phi + p\Lambda, \quad (31)$$

where  $\Lambda$  is some positive map. For the largest value of  $p$ ,  $p_*$ , for which the map  $\tilde{\Lambda}[p]$  is CP we obtain a completely positive approximation of  $\Lambda$ . Our aim now is to check if for all  $\Phi$ ,  $\tilde{\Lambda}[p_*]$  is EB. Notice, that through the C-J isomorphism the same approximation as in (31) is defined for EWs.

At the beginning let us consider the cases of two-qubits and qubit-qutrit. For any optimal witness  $W = Q^\Gamma$  with  $Q \geq 0$  it is then clear that the matrix

$$\tilde{W}(p) = (1-p)\sigma_{\text{sep}} + pQ^\Gamma \quad (0 \leq p \leq 1) \quad (32)$$

represents a separable state for any value of  $p$  for which  $\tilde{W}(p) \geq 0$ . This is because, as one may easily check,  $[\tilde{W}(p)]^\Gamma$  is positive for any  $p$ , and therefore, due to [6],  $\tilde{W}(p)$  is separable. It is then tempting to ask whether similar results hold for higher-dimensional systems. In what follows we show that this is not the case. For this purpose we introduce a simple one-parameter class of entanglement breaking channels corresponding to the known Werner states [8]. We then show that while the CP approximations for the transposition and reduction maps constructed using these channels are always EB, there exist a range of the parameter for which CP approximation of the Breuer-Hall map [43, 44] is no longer an EB channel.

Let us first introduce a completely positive map  $R_+(X) = [1/(m+1)][\text{Tr}(X)\mathbb{1}_m + X]$ . From both maps  $R_\pm$  (see Sec. 3.2 for the definition of  $R_-$ ) we construct the following channel

$$\Phi[\mu] = T \circ [\mu R_- + (1-\mu)R_+] \quad (0 \leq \mu \leq 1). \quad (33)$$

For  $\mu = 1/2$  the above map reduces to the fully depolarizing channel, i.e.,  $\Phi[1/2] = D$ . Via the C-J isomorphism,  $\Phi[\mu]$  corresponds to the so-called Werner states [8]:

$$\varrho_W(\mu) = \frac{2\mu}{m(m-1)}P_a + \frac{2(1-\mu)}{m(m+1)}P_s \quad (0 \leq \mu \leq 1), \quad (34)$$

where, as previously,  $P_a$  and  $P_s$  denote projectors onto antisymmetric and symmetric subspaces of  $\mathcal{H}_m$ , respectively. These states are separable iff  $\mu \leq 1/2$  and consequently for this range of  $\mu$  the map  $\Phi[\mu]$  represents an EB channel. Moreover, for  $\mu \in (0, 1)$ ,  $\varrho_W(\mu)$  is of full rank and therefore  $\Phi[\mu]$  is a good candidate for the construction of completely positive approximations of positive maps according to (31).

We can now test if such an approximations constructed using the map  $\Phi[\mu]$  give EBCs. Let us start from the transposition map. From (31) we have

$$\tilde{T}[p, \mu] = (1-p)\Phi[\mu] + pT \quad (0 \leq p \leq 1). \quad (35)$$

Partial application of the above map to the maximally entangled state  $P_m^+$  gives

$$\tilde{W}_T(p, \mu) = (1-p)\varrho_W(\mu) + \frac{p}{m}V_m, \quad (36)$$

where  $V_m = m[P_m^+]^\Gamma$ . It immediately follows (all the detailed calculations for this and the following cases may be found in Appendix A) that  $\tilde{W}_T(p, \mu) \geq 0$  if and only if

$$p \leq \frac{2\mu}{2\mu + m - 1}. \quad (37)$$

Then, it is easy to see that for  $p = p_*$ , the state  $\widetilde{W}_T(p_*, \mu)$  is separable for  $\mu \in (0, 1/2]$ . Thus, the approximation of the transposition map (35) build using  $\Phi[\mu]$  is also an EBC for any  $\mu \in (0, 1/2]$ .

Now, let us test an approximation (31) for next optimal decomposable map, the reduction map  $R_-$ . After mixing it with  $\Phi[\mu]$ , one gets  $\widetilde{R}_-[p, \mu] = (1-p)\Phi[\mu] + pR_-$  ( $0 \leq p \leq 1$ ). Via the C-J isomorphism we arrive at the matrix

$$\widetilde{W}_r(p, \mu) = (1-p)\varrho_W(\mu) + \frac{p}{m(m-1)}(\mathbb{1}_m \otimes \mathbb{1}_m - mP_m^+). \quad (38)$$

It follows (see Appendix A) that  $\widetilde{W}_r(p, \mu) \geq 0$  iff

$$p \leq \frac{2(1-\mu)}{3+m-2\mu}. \quad (39)$$

Then, for the threshold value of  $p$ ,  $\widetilde{W}_r(p_*, \mu)$  is separable for  $\mu \in (0, 1/2]$  and therefore again the CP approximation of  $R_-$  constructed with the aid of  $\Phi[\mu]$  is EB.

So far, we have studied only decomposable maps. Let us now consider an example of an optimal indecomposable positive map. One of the commonly known examples is the so-called Breuer-Hall map [43, 44] (notice that this map was recently generalized in Refs. [20, 21]) defined as

$$\Lambda_U(X) = \frac{1}{m-2} [\text{Tr}(X)\mathbb{1}_m - X - UX^T U^\dagger], \quad (40)$$

where now  $m$  is even,  $U^T = -U$ , and  $U^\dagger U = \mathbb{1}_m$ .

Partial application of the approximation of  $\Lambda_U$ ,  $\widetilde{\Lambda}_U[p, \mu] = (1-p)\Phi[\mu] + p\Lambda_U$  to  $P_m^+$  gives

$$\begin{aligned} \widetilde{\mathcal{W}}(p, \mu) &= \frac{1}{m} \left[ \left( \frac{2\mu(1-p)}{m-1} + \frac{p}{m-2} \right) P_a + \left( \frac{2(1-\mu)(1-p)}{m+1} + \frac{p}{m-2} \right) P_s \right] \\ &\quad - \frac{p}{m-2} P_m^+ - \frac{p}{m(m-2)} V_U, \end{aligned} \quad (41)$$

where  $V_U = (\mathbb{1}_m \otimes U)V(\mathbb{1}_m \otimes U^\dagger)$ . One sees that  $\widetilde{\mathcal{W}}(p, \mu) \geq 0$  iff

$$p \leq \frac{2(1-\mu)}{2(1-\mu) + m + 1}. \quad (42)$$

It then follows that for  $p = p_*$  the partial transposition of  $\widetilde{\mathcal{W}}(p_*, \mu)$  is not positive whenever  $\mu < [(m-1)(m-2)]/[2(m^2-2)]$ , meaning that for  $m \geq 3$ ,  $\widetilde{\mathcal{W}}(p_*, \mu)$  is not always separable and therefore at least for the above values of  $\mu$  the CP approximation (31) of  $\Lambda_U$  is certainly not EB. As a result the conjecture does not hold for any EBC  $\Phi$  such that  $(I \otimes \Phi)(P_m^+)$  is of full rank.

#### 4.2. Relaxing the conjecture

As shown in the preceding section, the conjecture does not have to hold when instead of  $D$ , one uses some other EB channel  $\Phi$  such that  $(I \otimes \Phi)(P_m^+)$  is of full rank. Nevertheless, the above examples show that at least in the case of the transposition and reduction maps, the approximations constructed from the channel  $\Phi[\mu]$  give EB map. It is then tempting to conjecture that for any positive map  $\Lambda$  there exists at least one EB channel such that the approximation of  $\Lambda$  with  $D$  substituted by this EB channel is again EB. Here we show that this statement is true for any positive  $\Lambda$ .

In order to proceed more formally let us consider a pure entangled state  $|\psi\rangle \in \mathcal{H}_{mn}$  and the set of states

$$S_\psi = \{\varrho \in \mathcal{D}_{m,n} | \langle \psi | \varrho | \psi \rangle = 0\}. \quad (43)$$

In general, the set  $S_\psi$  may contain all types of states with respect to separability, i.e., NPT and PPT entangled as well as separable states. Let us show that the latter constitute a subset of  $S_\psi$  with nonempty interior in  $S_\psi$ .

**Lemma.** *The subset of separable states in  $S_\psi$  has nonempty interior in  $S_\psi$ .*

**Proof.** For the sake of simplicity and clearness we assume that  $m = n$  and that Schmidt rank of  $|\psi\rangle$  is  $m$ . Then, since  $|\psi\rangle$  is related to  $|\psi_m^+\rangle$  via  $|\psi\rangle = A \otimes \mathbb{1}_m |\psi_m^+\rangle$  with full rank  $A$ , it suffices prove the above statement for  $|\psi\rangle = |\psi_m^+\rangle$ . It is nevertheless clear that the proof is also valid when  $m \neq n$  and the Schmidt rank of  $|\psi\rangle$  is less than  $\min\{m, n\}$ .

We will show, following the proof of theorem 1 from Ref. [54], that not all the separable states from  $S \equiv S_{\psi_m^+}$  can be approached arbitrarily close by entangled states from  $S$ . First, however, let us make some statements about the subset of pure separable states in  $S$ , denoted henceforward by  $S_{\text{sep}}$ . It follows from (43) that they satisfy the relation  $\langle e, f | \psi_m^+ \rangle = 0$ , which after simple calculations, implies that elements of  $S_{\text{sep}}$  are those product vectors for which  $|f\rangle \in \mathbb{C}^m$  is any vector orthogonal to  $|e^*\rangle$ . It is clear that they span  $(m^2 - 1)$ -dimensional subspace in  $\mathcal{H}_m$ , denoted by  $\tilde{\mathcal{H}}$ , while projectors onto these vectors span  $(m^2 - 1)^2$ -dimensional subspace in the Hilbert space of  $m^2 \times m^2$  matrices with the Hilbert-Schmidt scalar product, denoted  $\mathcal{V}_{\text{sep}}$ .

We are now ready to proceed with the main part of the proof. Let us take a separable state  $\sigma_{\text{sep}} \in S_\psi$  being a continuous convex combination over all product states from  $S_{\text{sep}}$ , i.e.,

$$\sigma_{\text{sep}} = \sum_{|e,f\rangle \in S_{\text{sep}}} p(e,f) |e,f\rangle \langle e,f|, \quad \sum_{|e,f\rangle \in S_{\text{sep}}} p(e,f) = 1, \quad (44)$$

where  $p(e,f) > 0$  for any  $|e,f\rangle \in S_{\text{sep}}$ , and assume that there exists a sequence of entangled states  $\varrho_n \in S_\psi$  convergent to  $\sigma_{\text{sep}}$  in the Hilbert-Schmidt norm. Then, it follows from Ref. [6] that for any  $\varrho_n$  there exists an EW  $W_n$  detecting it, i.e., such that  $\text{Tr}(W_n \varrho_n) < 0$ .

We can obviously write  $W_n = O_n + O_n^\perp$ , where  $O_n \in \mathcal{V}_{\text{sep}}$  and  $O_n^\perp$  belongs to the subspace of  $M_{m,m}$  orthogonal to  $\mathcal{V}_{\text{sep}}$  and spanned by  $P_m^+$ ,  $|\phi\rangle \langle \psi_m^+|$ , and  $|\psi_m^+\rangle \langle \phi|$  with arbitrary  $|\phi\rangle \in \tilde{\mathcal{H}}$ . Since  $|\psi_m^+\rangle \in \ker(\varrho_n)$ , it is clear that for any  $n$ ,  $\text{Tr}(W_n \varrho_n) = \text{Tr}(O_n \varrho_n) < 0$  and hence  $O_n$  are certainly nonzero matrices. Consequently, we can introduce normalized matrices  $\tilde{O}_n = O_n / \|O_n\|_2$  which now belong to the compact set of matrices from the unit sphere in  $M_{m,m}$ . As a result, there exists a subsequence  $\tilde{O}_{n_k}$  converging to some element  $\tilde{O} \in \mathcal{V}_{\text{sep}}$  with  $\|\tilde{O}\|_2 = 1$ . Then, analogous estimation to the one in [54] gives the following chain of inequalities

$$0 \leq \text{Tr}(\tilde{O}_{n_k} \sigma_{\text{sep}}) \leq \|\sigma_{\text{sep}} - \varrho_{n_k}\|_2, \quad (45)$$

which immediately implies that

$$\text{Tr}(\tilde{O} \sigma_{\text{sep}}) = 0. \quad (46)$$

On the other hand,  $\langle e, f | W_n | e, f \rangle = \langle e, f | O_n | e, f \rangle \geq 0$  for any  $|e, f\rangle \in S_{\text{sep}}$ , which, due to the continuity of the scalar product, yields

$$\langle e, f | \tilde{O} | e, f \rangle \geq 0 \quad (|e, f\rangle \in S_{\text{sep}}). \quad (47)$$

The latter together with (46) imply that  $\langle e, f | \tilde{O} | e, f \rangle = 0$  for any  $|e, f\rangle \in S_{\text{sep}}$ , which, due to the fact that  $\tilde{O} \in \mathcal{V}_{\text{sep}}$ , allows to infer that  $\tilde{O}$  is a zero matrix. This, however, leads to a contradiction with the fact that  $\|\tilde{O}\|_2 = 1$ . ■

Now we are prepared to proceed with our theorem.

**Theorem 4.** *Let  $W$  be some EW acting on  $\mathcal{H}_{m,n}$ . Then there exists a separable state  $\Sigma_{\text{sep}}$  such that the approximation of  $W$  constructed using  $\Sigma_{\text{sep}}$  according to (31) is separable. In other words, for any  $\Lambda$  there exists an EBC  $\Phi$  such that the CP approximation of  $\Lambda$  constructed using  $\Phi$  is EB.*

**Proof.** First let us construct the standard SPA of  $W$ . Denoting by  $-\lambda$  and by  $|\psi\rangle$  the smallest eigenvalue of  $W$  and the corresponding eigenvector, respectively, its SPA reads  $\tilde{W} = [1/(1 + \lambda mn)](\lambda \mathbb{1}_{mn} + W)$ . Let us assume that the latter is entangled as otherwise there is nothing to prove.

Clearly  $\langle \psi | \tilde{W} | \psi \rangle = 0$  and therefore the SPA of  $W$  brings us to the set  $S_\psi$  (see (43)). Naturally, now we can use the above lemma. Precisely, we can take a separable state  $\sigma_{\text{sep}}$  from the interior of the set of separable states in  $S_\psi$ , and mix it with  $\tilde{W}$  so that the resulting density matrix is separable. Since  $|\psi\rangle \in \ker(\sigma_{\text{sep}})$ ,  $\sigma_{\text{sep}}$  does not cover the lowest eigenvalue of  $\tilde{W}$ . Therefore we can treat the maximally mixed state used previously to construct  $\tilde{W}$  together with  $\sigma_{\text{sep}}$  as a postulated separable state allowing to build the approximation of  $W$ .

Let us now proceed more formally. Due to the very definition of  $\sigma_{\text{sep}}$ , there exists  $q \in (0, 1)$  such that  $\varrho(q) = q\sigma_{\text{sep}} + (1 - q)\tilde{W}$  is a separable state. Let  $q_*$  denote a minimal value of the mixing parameter for which  $\varrho(q)$  is separable. Then, one can check that the state we can use to construct the nonstandard SPA of  $W$  is

$$\Sigma_{\text{sep}} = \frac{1 - q_*}{q_* + \lambda mn} \left( \lambda \mathbb{1}_m \otimes \mathbb{1}_n + \frac{1 + \lambda mn}{1 - q_*} q_* \sigma_{\text{sep}} \right). \quad (48)$$

For this purpose, let us consider the mixture  $\overline{W}(r) = (1 - r)\Sigma_{\text{sep}} + rW$ . From the facts that the lowest eigenvalue of  $W$  is  $-\lambda$  and  $\langle \psi | \sigma_{\text{sep}} | \psi \rangle = 0$ , we see that  $\overline{W}(r)$  is positive for  $r \leq (1 - q_*)/(1 + \lambda mn)$ . It remains to show that for the threshold value  $r_* = (1 - q_*)/(1 + \lambda mn)$ ,  $\overline{W}(r_*)$  is separable. Direct check shows that  $\overline{W}(r_*)$  is exactly  $\varrho(q_*)$ , which we know to be separable. This finishes the proof. ■

Let us now comment on the obtained result. It is clear that if the set  $S_\psi$  consisted only of separable states, the conjecture would be immediately proven. However, in general this is not the case. Consequently, one way to prove it is to show that mixing given witness with maximally mixed state we always "land" in the set of separable states belonging to  $S_\psi$ .

#### 4.3. The conjecture in continuous-variable systems

In the present section we propose an extension of the notion of structural physical approximation onto the case of Gaussian positive maps, and we study the conjecture for the transposition map. First, however, we need to remind some basic concepts and notions from the theory of Gaussian states, operators and maps.

The Hilbert space describing  $n$  harmonic oscillators ( $n$  modes) is  $\mathcal{H} = L^2(\mathbb{R}^n)$ . To the  $i$ th mode there are associated two canonical observables  $X_i$  and  $P_i$  ( $i = 1, \dots, n$ ).



All these operators fulfil the canonical commutation relations  $[X_k, P_l] = i\delta_{kl}$ . Every operator  $A$  acting on  $B(\mathcal{H})$  can be represented in the following form [47, 48]:

$$A = \pi^{-n} \int_{R^{2n}} d^{2n}\xi \chi_A(\xi) W_{-\xi}, \quad (49)$$

where  $W_\xi = \exp(-i\xi^T R)$  is the so-called Weyl operator (displacement operator),  $R = (X_1, P_1, \dots, X_n, P_n)$  and  $\chi_A(\xi) = \text{Tr}(AW_\xi)$  is the characteristic function of  $A$ . An operator  $A$ , whose characteristic function  $\chi_A(\xi)$  has the following form

$$\chi_A(\xi) = \exp\left(-\frac{1}{4}\xi^T \gamma \xi + i d^T \xi - c\right), \quad (50)$$

is called a *Gaussian operator* [51]. In the above  $\gamma$  and  $d$  denote a covariance-like matrix and a displacement-like vector with entries given by

$$\gamma_{ij} = 2\text{Re}\{\text{Tr}[(R_i - d_i)(R_j - d_j)A]\}, \quad d_i = \text{Tr}(R_i A), \quad (51)$$

and  $c$  stands for the normalization constant. Let the set of all Gaussian operators acting on  $\mathcal{H}$  be denoted by  $Q(\mathcal{H})$ . Among all the elements of  $Q(\mathcal{H})$  we distinguish those for which  $\gamma, d$ , and  $c$  are real and  $\gamma$  obeys  $\gamma + iJ_n \geq 0$  with  $J_n$  defined as

$$J_n = \bigoplus_{i=1}^n J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (52)$$

These elements are just Gaussian states, and the corresponding  $\gamma$ s are proper covariance matrices.

A linear map  $\Lambda : B(\mathcal{H}) \mapsto B(\mathcal{H}')$  that maps Gaussian operators to Gaussian operators, i.e.,  $\Lambda[Q(\mathcal{H})] \subseteq Q(\mathcal{H}')$  is called *Gaussian*. In a full analogy to the finite-dimensional case, one defines positive, completely positive (among them also quantum channels), and entanglement breaking Gaussian maps [49]. Let us only remind [53] that the action of any *Gaussian channel*  $\mathcal{G} : B(\mathcal{H}) \mapsto B(\mathcal{H})$  can be represented on the ground of covariance matrices as

$$\gamma' = X^T \gamma X + Y, \quad d' = X^T d + v, \quad (53)$$

where  $\gamma'$  ( $d'$ ) and  $\gamma$  ( $d$ ) are covariance matrices (displacement vectors) of the Gaussian states after and before an application of  $\mathcal{G}$ , respectively. Then,  $X$  and  $Y$  are some real  $2n \times 2n$  matrices, which in order to guarantee complete positivity of  $\mathcal{G}$  must obey

$$Y + iJ_n - iX^T J_n X \geq 0. \quad (54)$$

Then, it was shown in Ref. [49] that a given Gaussian channel is entanglement breaking if and only if  $Y$  can be decomposed as  $Y = A + B$  with real and symmetric  $A$  and  $B$  satisfying

$$A \geq -iJ_n, \quad B \geq -iX^T J_n X. \quad (55)$$

Having recalled all the necessary notions we can pass to the concept of SPA in Gaussian maps. To the best of our knowledge, the first attempt to get the best approximation (in terms of the fidelity) of the transposition map in continuous-variable systems was made in Ref. [12]. The authors looked for the best completely positive approximation (in terms of the fidelity) of the transposition map under the assumption that it has to work equally well for all pure states. As this cannot be achieved in the infinite-dimensional case, they proposed approximation which depends on the input states.

Here we follow a different approach, which is more consistent with the standard SPA (7) in the sense that it does not depend on the input state. In order to proceed more formally let  $\Lambda : B(\mathcal{H}) \mapsto B(\mathcal{H})$  be some positive Gaussian map. We know from Ref. [46] that on the level of CMs its action can be represented by some map which for simplicity we will also denote by  $\Lambda$ . Clearly, we cannot construct SPA as in Eq. (7) as addition of an identity to the gaussian density operator may easily throw us out of the set of density operators. Nevertheless, as we will see below, the above reasoning can successfully be adopted, at least in some cases, on the level of covariance matrices. This is because an addition of  $\mathbb{1}$  with some weight  $p$  to the covariance matrix  $\gamma' = \Lambda(\gamma)$ , i.e.,

$$\tilde{\Lambda}_p(\gamma) = \Lambda(\gamma) + p\mathbb{1}, \quad \tilde{\Lambda}_p(d) = \Lambda(d), \quad (56)$$

keeps us in the set of proper covariance matrices. On the level of density operators the above means composition of  $\Lambda$  with Gaussian channel  $\Phi[p]$  defined as

$$\Phi_p(\rho) = \frac{1}{\pi^n p^n} \int d^{2n}\xi W_\xi \rho W_\xi^\dagger e^{-\frac{\xi^T \xi}{p}}. \quad (57)$$

From the physical point of view, the latter adds classical gaussian noise with variance  $p$  to a given state [47]. Consequently, on the level of density matrices, our SPA is defined as  $\tilde{\Lambda}[p] = \Phi[p] \circ \Lambda$ .

Now, we see that if the positive Gaussian map  $\Lambda$  can be represented as in Eq. (53), one can always find  $p$  such that  $Y + p\mathbb{1} + iJ_n - iX^T J_n X \geq 0$ . Consequently, at least for any positive Gaussian map which action on covariance matrices is given by Eq. (53), the approximation defined by Eq. (56) leads to a completely positive map.

Let us now consider the conjecture in the context of the transposition map. We will show that the Gaussian SPA of transposition map leads to the entanglement breaking channel. To this aim recall that the action of transposition map on an  $n$ -mode Gaussian state (characterized by  $\gamma$  and  $d$ ) takes form [52]:

$$\gamma' = \Delta \gamma \Delta, \quad d' = \Delta d, \quad (58)$$

where

$$\Delta = \bigotimes_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (59)$$

From Eqs. (56) and (58) one infers that the action of SPA of the transposition map  $\tilde{T}[p] = \Phi[p] \circ T$  on the level of covariance matrices reads  $\tilde{T}[p](\gamma) = T(\gamma) + p\mathbb{1}$ . The condition (54) tells us that  $\tilde{T}[p]$  is completely positive iff

$$p\mathbb{1} + i \bigotimes_{i=1}^n \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \geq 0, \quad (60)$$

which is satisfied for  $p \geq 2$ . Let us now check if for the threshold value  $p_* = 2$ , the map  $\tilde{T}[2]$  is entanglement breaking. Indeed, one sees that in this case  $Y = 2\mathbb{1}$  can be decomposed as  $Y = A + B$  with  $A = B = \mathbb{1}$  and both matrices  $A$  and  $B$  fulfill the conditions (55).

Concluding, let us mention that the obtained channel in the one-mode case can be represented as

$$\tilde{T}_1(\rho) = \frac{1}{\pi} \int d^2\xi \langle \xi_x, \xi_p | \rho | \xi_x, \xi_p \rangle | \xi_x, -\xi_p \rangle \langle \xi_x, -\xi_p |, \quad (61)$$

where  $|\xi_x, \xi_p\rangle$  is a coherent state with the displacements given by  $\xi_x$  and  $\xi_p$ . In the general case of  $n$  modes we have  $T_n = \bigotimes_{i=1}^n T_1$ .

## 5. Conclusions

The structural physical approximation [9, 10, 11] is one of the known solutions to the problem of applicability of positive maps in an experimental detection of entanglement. The recent observations [17, 19, 20, 21] that SPAs of many known optimal positive maps give EB channels or maps in general, makes this notion interesting also from the mathematical point of view. Certainly, a full proof of this conjecture would shed new light on the geometry of convex sets of states with positive partial transpose and their relations to entanglement witnesses. Motivated by its mathematical importance, we have addressed several issues related to the conjecture.

Let us shortly summarize the obtained results. First, we have extended the set of optimal witnesses satisfying the conjecture. For instance, we have proven it for witnesses  $W = |\psi\rangle\langle\psi|^\Gamma$  for any entangled  $|\psi\rangle$ . Also, utilizing recent results of Ref. [41, 20], we have determined the whole class of decomposable witnesses satisfying the conjecture which as a particular case contain all witnesses  $W = Q^\Gamma$  with  $Q$  acting on the antisymmetric subspace of  $\mathbb{C}^m \otimes \mathbb{C}^m$  with the first subsystem in the maximally mixed state  $\mathbb{1}_m/m$ .

Then, we have considered SPAs constructed from other than the completely depolarizing channel  $D$  entanglement breaking channels. We have shown that there exist EBC which certainly does not satisfy the statement of the conjecture (see Sec. 2.3). In other words, there exists EB channels that can be used to construct SPAs, nevertheless, the obtained CP maps do not have to be EB. On the other hand, we have proven an interesting fact that for any positive map  $\Lambda$  there exists an EB map  $\Phi$  such that the SPA of  $\Lambda$  constructed with the help of  $\Phi$  is EB. Finally, we have asked a similar question in the case of continuous-variable systems. Precisely, we have proposed a generalization of the notion of SPA to the Gaussian case and proven that such SPA of the transposition map is entanglement breaking.

Let us just shortly sketch the possible lines of further investigations related to this subject. First of all, the complete proof of the conjecture is still missing. Then, studies on the relations of SPAs to the conjecture in the case of multipartite and continuous-variable systems seem interesting. We only touch this problem in CV systems and further works clarifying it are desirable. One knows, however, that the theory of positive maps serving for detection of entanglement is much better developed in finite-dimensional systems. This is certainly because the set of covariance matrices is convex and closed and therefore one can define entanglement witnesses on the level of covariance matrices instead of density operators (see e.g. [50]). Also, the notion of optimality of positive maps in this case is missing. Finally, following our results, one could investigate if other SPAs (than the one constructed from  $D$ ) that lead to the entanglement breaking channels are maybe easier to implement experimentally.

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**Appendix A. Detailed calculations for Sec. 4.1**

*Transposition map.* Due to the fact that  $V_m = P_s - P_a$ , we can rewrite (36) in the following form

$$\widetilde{W}_T(p, \mu) = \frac{1}{m} \left\{ \left[ \frac{2\mu(1-p)}{m-1} - p \right] P_a + \left[ \frac{2\mu(1-p)(1-\mu)}{m+1} + p \right] P_s \right\}. \quad (\text{A.1})$$

Since  $P_s P_a = 0$  it is clear that  $\widetilde{W}_T(p, \mu) \geq 0$  if and only if the expression appearing in the first square bracket is nonnegative. This means that  $\widetilde{T}[p, \mu]$  is completely positive for  $p$  obeying (37). For the threshold value of  $p$ , i.e.,  $p_* = 2\mu/(2\mu + m - 1)$ , the matrix  $\widetilde{W}(p_*, \mu)$  is just the normalized projector onto  $\mathcal{S}_m$  and therefore separable.

*Reduction map.* Utilizing the fact that  $P_s + P_a = \mathbb{1}_m$  one can rewrite Eq. (38) as

$$\widetilde{W}_r(p, \mu) = \frac{1}{m} \left\{ \left[ \frac{2\mu(1-p) + p}{m-1} \right] P_a + \left[ \frac{2(1-\mu)(1-p)}{m+1} + \frac{p}{m-1} \right] P_s \right\} - \frac{p}{m-1} P_m^+. \quad (\text{A.2})$$

Since  $|\psi_m^+\rangle \in \mathcal{S}_m$ ,  $W(p, \mu) \geq 0$  iff  $p$  satisfies (39). For the threshold value of  $p$ ,  $\widetilde{W}_r(p_*, \mu)$  takes the following form

$$\widetilde{W}_r(p_*, \mu) = N_1 \left[ \frac{1+m\mu}{m} P_{as} + (1-\mu)(P_{\text{sym}} - P_m^+) \right] \quad (\text{A.3})$$

with  $N_1 = 2/[(m-1)(3+m-2\mu)]$ . In the above one recognizes the so-called  $OO$ -invariant states, that is, density matrices which are invariant under bilateral application of the orthogonal group  $O(m)$ . These states were investigated in [42] and it was shown there that all  $OO$ -invariant states with positive partial transposition are separable. It then suffices to check then spectrum of the partial transpose of  $[\widetilde{W}_r(p_*, \mu)]$ . After some simple movements, the latter can be written as

$$[\widetilde{W}_r(p_*, \mu)]^\Gamma = N_2 [(m-1+2\mu)P_s + (m+3-2\mu)P_a + (m-2\mu m-1)mP_m^+], \quad (\text{A.4})$$

where  $N_2 = 1/[m(m-1)(m+3-2\mu)]$ . This implies that  $[\widetilde{W}_r(p_*, \mu)]^\Gamma \geq 0$  and therefore according to the results from [42],  $\widetilde{W}_r(p_*, \mu)$  is separable for any  $\mu \in (0, 1/2]$ .

*The Breuer-Hall map.* In order to determine the range of  $p$  for which  $\widetilde{W}(p, \mu)$  is positive we need to make the following observations. Denoting by  $|\psi_s\rangle$  ( $|\psi_a\rangle$ ) an arbitrary state belonging to  $\mathcal{S}_m$  ( $\mathcal{A}_m$ ), we see that  $V_U |\psi_s\rangle = -U \otimes U |\psi_s\rangle$  ( $V_U |\psi_a\rangle = U \otimes U |\psi_a\rangle$ ) for any real unitary satisfying  $U^T = -U$ . This implies that  $V_U \mathcal{S}_m \subseteq \mathcal{S}_m$  and similarly for  $\mathcal{A}_m$ . It then follows that the smallest eigenvalue of  $\widetilde{W}(p, \mu)$  corresponds to  $|\psi_m^+\rangle$  and  $\widetilde{W}(p, \mu)$  is positive iff  $p$  obeys (42). The partial transposition of  $\widetilde{W}(p_*, \mu)$  reads

$$[\widetilde{W}(p_*, \mu)]^\Gamma = N_3 \left[ \left( m - \frac{2\mu}{m-1} \right) \mathbb{1}_{m^2} - 2(1-\mu)V_m \left( m - 2 - \frac{2m\mu(m-2)}{m-1} \right) mP_m^+ - 2(1-\mu)m(\mathbb{1}_m \otimes U)P_m^+(\mathbb{1}_m \otimes U^\dagger) \right], \quad (\text{A.5})$$

where  $N_3$  is a constant irrelevant for further consideration. Since  $U = -U^T$ , it follows that  $\text{Tr}(U) = 0$  and therefore  $|\psi_m^+\rangle$  and  $\mathbb{1}_m \otimes U |\psi_m^+\rangle$  are orthogonal. Consequently, for

$$\mu < \frac{(m-1)(m-2)}{2(m^2-2)}, \quad (\text{A.6})$$

$[\widetilde{W}(p_*, \mu)]^\Gamma$  is not positive and hence  $\widetilde{W}(p_*, \mu)$  cannot be separable.

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